International Series of Numerical Mathematics, Vol. 123,

Integral inequalities for algebraic polynomials

Gradimir V. Milovanović

Abstract

In this paper we consider two extremal problems for algebraic polynomials in L^2 metrics.

(1) Let $\mathcal{P}_n$ be the class of all algebraic polynomials $P(x) = \sum_{k=0}^{n} a_k x^k$ of degree at most n and $\|P\|_{d\sigma} = (\int_{\mathbb{R}} |P(x)|^2 d\sigma(x))^{1/2}$, where $d\sigma(x)$ is a nonnegative measure on $\mathbb{R}$. We determine the best constant in the inequality $|a_k| \le C_{n,k}(d\sigma)\|P\|_{d\sigma}$, for $k = 0, 1, \ldots, n$, when $P \in \mathcal{P}_n$ and such that $P(\xi_k) = 0$, $k = 1, \ldots, m$. The cases $C_{n,n}(d\sigma)$ and $C_{n,n-1}(d\sigma)$ were studied by Milovanović and Guessab [5], and only for the Legendre measure by Tariq [9].

(2) Let $\hat{\mathcal{P}}_N$ be the set of all monic algebraic polynomials of degree N and ε_s be Mth roots of unity, i.e., $\varepsilon_s = \exp(i2\pi s/M)$, $s = 0, 1, \ldots, M-1$. Polynomials orthogonal on the radial rays in the complex plane with respect to the inner product

$$(f, g) = \int_0^a \left(\sum_{s=0}^{M-1} f(x\varepsilon_s)\overline{g(x\varepsilon_s)} \right) w(x)\, dx$$

have been introduced and studied recently in [3]. Here, w is a weight function and $0 < a \le +\infty$. We consider the extremal problem

$$\inf_{P \in \hat{\mathcal{P}}_N} \int_0^a \left(\sum_{s=0}^{M-1} |P(x\varepsilon_s)|^2 \right) w(x)\, dx,$$

as well as some inequalities for coefficients of polynomials under some restrictions of the polynomial class.

1. Introduction

Let $\mathcal{P}_n$ be the class of algebraic polynomials $P(x) = \sum_{k=0}^{n} a_k x^k$ of degree at most n. The first inequality of the form $|a_k| \le C_{n,k}\|P\|$ was given by V.A. Markov [2]. Namely, if

$$\|P\| = \|P\|_\infty = \max_{x\in[-1,1]} |P(x)|$$

and $T_n(x) = \sum\limits_{\nu=0}^{n} t_{n,\nu}x^\nu$ denotes the n-th Chebyshev polynomial of the first kind, Markov proved that

$$|a_k| \le \begin{cases} |t_{n,k}| \cdot \|P\|_\infty & \text{if } n-k \text{ is even,} \\ |t_{n-1,k}| \cdot \|P\|_\infty & \text{if } n-k \text{ is odd.} \end{cases}$$

Precisely,

$$|a_k| \le \frac{2^{k-1} D_{n,k}}{k!} \|P\|_\infty, \tag{1.1}$$

where

$$D_{n,k} = \begin{cases} n \dfrac{\left(\frac{n+k-2}{2}\right)!}{\left(\frac{n-k}{2}\right)!} & \text{if } n-k \text{ is even,} \\ (n-1) \dfrac{\left(\frac{n+k-3}{2}\right)!}{\left(\frac{n-k-1}{2}\right)!} & \text{if } n-k \text{ is odd.} \end{cases}$$

For $k = n$ (1.1) reduces to the well-known Chebyshev inequality

$$|a_n| \le 2^{n-1} \|P\|_\infty. \tag{1.2}$$

Using a restriction on the polynomial class, inequality (1.1) could be improved. For example, taking $P(1) = 0$ or $P(-1) = 0$, Schur [8] found the following improvement of (1.2)

$$|a_n| \le 2^{n-1} \left(\cos \frac{\pi}{4n}\right)^{2n} \|P\|_\infty.$$

This result was extended later by Rahman and Schmeisser [7] for polynomials with real coefficients, which have at most $n-1$ distinct zeros in $(-1, 1)$.

Our interest are the corresponding inequalities in L^2 norm. Such one result was obtained by Labelle [1]

$$|a_k| \le \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{k!} \left(k + \frac{1}{2}\right)^{1/2} \binom{[(n-k)/2] + k + 1/2}{[(n-k)/2]} \|P\|_2, \tag{1.3}$$

where $P \in \mathcal{P}_n$, $0 \le k \le n$,

$$\|P\| = \|P\|_2 = \left(\int_{-1}^{1} |P(x)|^2 \, dx\right)^{1/2},$$

and the symbol $[x]$ denotes the integral part of x. Equality in this case is attained only for the constant multiples of the polynomial

$$\sum_{\nu=0}^{[(n-k)/2]} (-1)^\nu (4\nu + 2k + 1) \binom{k + \nu - 1/2}{\nu} P_{k+2\nu}(x),$$

where $P_m(x)$ denotes the Legendre polynomial of degree m.

This result was improved by Tariq [9]. Under restriction $P(1)=0$, he proved that

$$|a_n| \leq \frac{n}{n+1}\cdot\frac{(2n)!}{2^n(n!)^2}\left(\frac{2n+1}{2}\right)^{1/2}\|P\|_2, \tag{1.4}$$

with equality case

$$P(x) = P_n(x) - \frac{1}{n^2}\sum_{\nu=0}^{n-1}(2\nu+1)P_\nu(x).$$

In comparing with (1.4), we note that in inequality (1.3) for $k=n$ the factor $n/(n+1)$ does not exist. Tariq also obtained that

$$|a_{n-1}| \leq \frac{(n^2+2)^{1/2}}{n+1}\cdot\frac{(2n-2)!}{2^{n-1}((n-1)!)^2}\left(\frac{2n-1}{2}\right)^{1/2}\|P\|_2, \tag{1.5}$$

with equality case

$$P(x) = \frac{2n+1}{n^2+2}P_n(x) - P_{n-1}(x) + \frac{1}{n^2+2}\sum_{\nu=0}^{n-2}(2\nu+1)P_\nu(x).$$

In the absence of the hypothesis $P(1)=0$ the factor $(n^2+2)^{1/2}/(n+1)$ appearing on the right-hand side of (1.5) is to be dropped.

Recently, Milovanović and Guessab [5] have extended this result to polynomials with real coefficients, which have m zeros on real line.

In this paper we consider more general problem including L^2 norm of polynomials with respect to a nonnegative measure on the real line $\mathbb{R}$ and we give estimates for all coefficients. Also, we consider extremal problems for polynomials with respect to an inner product defined on the radial rays in the complex plane. Polynomials orthogonal with respect to such a product have been introduced and studied in [3].

2. Estimates for coefficients

For $\xi_k \in \mathbb{C}$, $k=1,\ldots,m$, we consider a restricted polynomial class

$$\mathcal{P}_n(\xi_1,\ldots,\xi_m) = \{P\in\mathcal{P}_n \mid P(\xi_k)=0,\ k=1,\ldots,m\} \qquad (0\leq m\leq n).$$

In the case $m=0$ this class of polynomials reduces to $\mathcal{P}_n$. If $\xi_1=\cdots=\xi_k=\xi$ $(1\leq k\leq m)$ then the restriction on polynomials at the point $x=\xi$ becomes $P(\xi)=P'(\xi)=\cdots=P^{(k-1)}(\xi)=0$.

Let

$$\prod_{i=1}^{m}(x-\xi_i)=x^m-s_1x^{m-1}+\cdots+(-1)^{m-1}s_{m-1}x+(-1)^m s_m$$

where s_k denotes elementary symmetric functions of $\xi_1,\ldots,\xi_m$, i.e.,

$$s_k=\sum \xi_1\cdots\xi_k \quad \mathit{for} \quad k=1,\ldots,m. \tag{2.1}$$

For $k=0$ we have $s_0=1$, and $s_k=0$ for $k>m$ or $k<0$.

Let $d\sigma(x)$ be a given nonnegative measure on the real line $\mathbb{R}$, with compact or infinite support, for which all moments $\mu_k=\int_{\mathbb{R}} x^k\,d\sigma(x)$, $k=0,1,\ldots$, exist and are finite, and $\mu_0>0$. In that case, there exist a unique set of orthonormal polynomials $\pi_n(\cdot)=\pi_n(\cdot;d\sigma)$, $n=0,1,\ldots$, defined by

$$\pi_n(x)=b_n^{(n)}(d\sigma)x^n+b_{n-1}^{(n)}(d\sigma)x^{n-1}+\cdots+b_0^{(n)}(d\sigma), \qquad b_n^{(n)}(d\sigma)>0,$$
$$(\pi_n,\pi_m)=\delta_{nm}, \qquad n,m\ge 0,$$

where

$$(f,g)=\int_{\mathbb{R}} f(x)\overline{g(x)}\,d\sigma(x) \qquad (f,g\in L^2(\mathbb{R})). \tag{2.2}$$

Also, we put

$$\|P\|_{d\sigma}=\sqrt{(P,P)}=\left(\int_{\mathbb{R}}|P(x)|^2\,d\sigma(x)\right)^{1/2}. \tag{2.3}$$

We mention first that every polynomial $P(x)=\sum\limits_{\nu=0}^{n} a_\nu x^\nu\in\mathcal{P}_n$ can be represented in the form

$$P(x)=\sum_{\nu=0}^{n}\alpha_\nu\pi_\nu(x;d\sigma),$$

where $\alpha_\nu=(P,\pi_\nu)$, $\nu=0,1,\ldots,n$, and the inner product is given by (2.2). Then we have

$$a_{n-k}=\sum_{i=0}^{k}\alpha_{n-i}b_{n-k}^{(n-i)}(d\sigma)=\left(P,\sum_{i=0}^{k}b_{n-k}^{(n-i)}(d\sigma)\pi_{n-i}\right), \quad k=0,1,\ldots,n, \tag{2.4}$$

where $\pi_\nu(\cdot)=\pi_\nu(\cdot;d\sigma)$. If we suppose that $P\in\mathcal{P}_n(\xi_1,\ldots,\xi_m)$, then we have

$$P(x)=Q(x)\prod_{k=1}^{m}(x-\xi_k), \tag{2.5}$$

where $Q(x)=a'_{n-m}x^{n-m}+a'_{n-m-1}x^{n-m-1}+\ldots+a'_0\in\mathcal{P}_{n-m}$. Also, we have

$$\prod_{i=1}^{m}(x-\xi_i)=x^m-s_1x^{m-1}+\cdots+(-1)^{m-1}s_{m-1}x+(-1)^m s_m$$

where s_k, $k = 0, 1, \ldots, m$, denotes elementary symmetric functions (2.1). Using (2.5) we find that

$$a_{n-k} = \sum_{i=0}^{k} a'_{n-m-i}(-1)^{k-i} s_{k-i}, \qquad k = 0, 1, \ldots, n,$$

and $a'_k = 0$ for $k < 0$ and $k > n - m$. Now, the corresponding equalities (2.4) for polynomial Q in the measure $d\hat{\sigma}(x)$, given by

$$d\hat{\sigma}(x) = \prod_{k=1}^{m} |x - \xi_k|^2 d\sigma(x) \tag{2.6}$$

become

$$a'_{n-m-i} = \Bigl(Q, \sum_{j=0}^{i} \hat{b}_{n-m-i}^{(n-m-j)} \hat{\pi}_{n-m-j}\Bigr), \quad i = 0, 1, \ldots, n-m,$$

where $\hat{\pi}_\nu(\cdot) = \pi_\nu(\cdot; d\hat{\sigma})$. According to (2.5), we have

$$a_{n-k} = \sum_{i=0}^{k} (-1)^{k-i} s_{k-i} \left(Q, \sum_{j=0}^{i} \hat{b}_{n-m-i}^{(n-m-j)} \hat{\pi}_{n-m-j} \right) = (Q, W_{n-m}) \tag{2.7}$$

where

$$\begin{aligned} W_{n-m}(x) &= \sum_{i=0}^{k} (-1)^{k-i} s_{k-i} \sum_{j=0}^{i} \hat{b}_{n-m-i}^{(n-m-j)} \hat{\pi}_{n-m-j}(x) \\ &= \sum_{j=0}^{k} \hat{\pi}_{n-m-j}(x) \sum_{i=j}^{k} (-1)^{k-i} s_{k-i} \hat{b}_{n-m-i}^{(n-m-j)} \end{aligned}$$

and $\hat{b}_\nu^{(\mu)} = 0$ for $\nu < 0$.

Now, we can prove the following result:

Theorem 2.1. *Let $P \in \mathcal{P}_n(\xi_1, \ldots, \xi_m)$ and $s_1, \ldots, s_m$ be given by (2.1). If the measure $d\hat{\sigma}(x)$ is given by (2.6) and $\|P\|_{d\sigma}$ is defined by (2.3), then*

$$|a_{n-k}| \le \left(\sum_{j=0}^{k} \left(\sum_{i=j}^{k} (-1)^{k-i} s_{k-i} \hat{b}_{n-m-i}^{(n-m-j)} \right)^2 \right)^{1/2} \|P\|_{d\sigma}, \tag{2.8}$$

for $k = 0, 1, \ldots, n$, where $\hat{b}_\nu^\mu = b_\nu^\mu(d\hat{\sigma})$, $\nu = 0, 1, \ldots, \mu$, are the coefficients in the orthonormal polynomial $\hat{\pi}_\mu(\cdot) = \pi_\mu(\cdot; d\hat{\sigma})$. Inequality (2.8) is sharp and becomes *an equality if and only if $P(x)$ is a constant multiple of the polynomial*

$$\left(\sum_{j=0}^{k}\hat{\pi}_{n-m-j}(x)\sum_{i=j}^{k}(-1)^{k-i}s_{k-i}\hat{b}_{n-m-i}^{(n-m-j)}\right)\prod_{k=1}^{m}(x-\xi_k).$$

Proof. Using Cauchy inequality from (2.7) we get

$$|a_{n-k}| \le C_{n,n-k}\|Q\|_{d\hat{\sigma}}$$

where

$$C_{n,n-k} = \|W_{n-m}\|_{d\hat{\sigma}} = \left(\sum_{j=0}^{k}\left(\sum_{i=j}^{k}(-1)^{k-i}s_{k-i}\hat{b}_{n-m-i}^{(n-m-j)}\right)^2\right)^{1/2}.$$

Since

$$\|Q\|_{d\hat{\sigma}}^2 = \int_{\mathbb{R}}|Q(x)|^2 d\hat{\sigma}(x) = \int_{\mathbb{R}}|P(x)|^2 d\sigma(x) = \|P\|_{d\sigma}^2$$

we obtain inequality (2.8). The extremal polynomial is

$$x \mapsto W_{n-m}(x)\prod_{k=1}^{m}(x-\xi_k). \qquad \square$$

For $k=0$ and $k=1$ this theorem gives the results obtained by Milovanović and Guessab [5] (see also [6], pp. 432–439). In the simplest case when $d\sigma(t)=dt$ on $(-1,1)$, these results reduce to the Tariq's inequalities (1.4) and (1.5).

3. Extremal problems on the radial rays

Let $\hat{\mathcal{P}}_N$ be the set of all monic algebraic polynomials of degree N and ε_s be Mth roots of unity, i.e., $\varepsilon_s = \exp(i2\pi s/M)$, $s=0,1,\ldots,M-1$. Polynomials orthogonal on the radial rays with respect to the inner product

$$(f,g) = \int_0^a\left(\sum_{s=0}^{M-1} f(x\varepsilon_s)\overline{g(x\varepsilon_s)}\right)w(x)\,dx \tag{3.1}$$

have been introduced and studied recently in [3]. Here, w is a weight function and $0<a\le+\infty$. The case $a=+\infty$ with an exponential weight gives the generalized Hermite polynomials on the radial rays (see [4]).

In this section we consider the following extremal problem

$$\inf_{P\in\hat{\mathcal{P}}_N}\int_0^a \left(\sum_{s=0}^{M-1} |P(x\varepsilon_s)|^2\right) w(x)\,dx. \tag{3.2}$$

Theorem 3.1. *Let $0 < a \le +\infty$. For each $P \in \hat{\mathcal{P}}_N$ we have*

$$\int_0^a \left(\sum_{s=0}^{M-1} |P(x\varepsilon_s)|^2\right) w(x)\,dx \ge \|\pi_N\|^2, \tag{3.3}$$

where (π_N) are monic orthogonal polynomials with respect to the inner product (3.1). *In* (3.3) *equality holds if and only if $P(z) = \pi_N(z)$.*

Proof. Let $P \in \hat{\mathcal{P}}_N$ and $(\pi_k)_{k=0}^{+\infty}$ be a sequence of the monic polynomials orthogonal with respect to the inner product (3.1), where $0 < a \le +\infty$. Then $P(z)$ can be expressed in the form

$$P(z) = \sum_{k=0}^{N} c_k \pi_k(z),$$

where $c_k = (P, \pi_k)/\|\pi_k\|^2$, $k = 0, 1, \ldots, N$. Notice that $c_N = 1$. Since

$$\|P\|^2 = \sum_{k=0}^{N} |c_k|^2 \|\pi_k\|^2 \ge |c_N|^2 \|\pi_N\|^2 = \|\pi_N\|^2,$$

we obtain (3.3), with equality if and only if $P(z) = \pi_N(z)$. □

Using an explicit form of the norm in a particular case when $M = 4$ (see [3], Remark 5.4) we obtain the following result:

Corollary 3.2. *Let $P \in \hat{\mathcal{P}}_N$, $N = 4n + \nu$, $n = [N/4]$, and $\nu \in \{0, 1, 2, 3\}$. Then*

$$\int_0^1 \Big(|P(x)|^2 + |P(ix)|^2 + |P(-x)|^2 + |P(-ix)|^2\Big) dx \ge L_N,$$

where

$$L_N = \frac{4}{2N+1}\left(\prod_{k=n}^{2n-1} \frac{4(k-\nu+1)}{4k+2\nu+1}\right)^2$$

for $N \ge 4$, and $L_N = 4/(2N+1)$ if $N \le 3$.

Taking an arbitrary polynomial P of degree N, i.e., $P \in \mathcal{P}_N$, and using Theorem 3.1 we can obtain an estimate for its leading coefficient a_N like (1.2) or (1.3) without the factor $n/(n+1)$.

Corollary 3.3. *Let* $0 < a \le +\infty$. *For each* $P_N(z) = \sum\limits_{k=0}^{N} a_k z^k \in \mathcal{P}_N$ *we have*

$$|a_N| \le \frac{\|P\|}{\|\pi_N\|}, \tag{3.4}$$

where (π_N) *are monic orthogonal polynomials with respect to the inner product* (3.1). *In* (3.4) *equality holds if and only if* $P(z)$ *is a constant multiple of the polynomial* $\pi_N(z)$.

We consider now a restricted class of polynomials of degree at most N such that $P(z)$ has a multiple zero of the order m in the origin $z = 0$. Denote this class by $\mathcal{P}_N^m$. Evidently, each polynomial $P \in \mathcal{P}_N^m$ can be expressed in the form $P(z) = z^m Q(z)$, where $Q \in \mathcal{P}_{N-m}$.

Introduce the weight function $x \mapsto w_m(x) = x^m w(x)$ on $(0, a)$, where $m \in \mathbb{N}$, and let $(\pi_{k,m})_{k=0}^{+\infty}$ be the set of monic polynomials orthogonal with respect to the inner product (3.1) with the weight function w_m instead of w. Then Corollary 3.3 can be interpreted in the following way:

Corollary 3.4. *Let* $0 < a \le +\infty$ *and* P *be an arbitrary polynomial in* $\mathcal{P}_N^m$. *Then for its leading coefficient the following inequality*

$$|a_N| \le \frac{\|P\|}{\|\pi_{N-m,m}\|}$$

holds, with equality if and only if $P(z)$ *is a constant multiple of the polynomial* $z^m \pi_{N-m,m}(z)$.

References

[1] G. Labelle, *Concerning polynomials on the unit interval.* Proc. Amer. Math. Soc. **20** (1969), 321–326.

[2] V.A. Markov, *On functions deviating least from zero in a given interval.* Izdat. Imp. Akad. Nauk, St. Petersburg 1892 (Russian) [German transl. Math. Ann. **77** (1916), 218–258].

[3] G.V. Milovanović, *A class of orthogonal polynomials on the radial rays in the complex plane.* J. Math. Anal. Appl., to appear.

[4] G.V. Milovanović, *Generalized Hermite polynomials on the radial rays in the complex plane.* In: Theory of Functions and Applications (H.B. Nersessian, ed.), Collection of Works Dedicated to the Memory of M.M. Djrbashian, Louys Publishing House, Yerevan, 1995, pp. 125–129.

[5] G.V. Milovanović and A. Guessab, *An estimate for coefficients of polynomials in L^2 norm.* Proc. Amer. Math. Soc. **120** (1994), 165–171.

[6] G.V. Milovanović, D. S. Mitrinović and Th. M. Rassias, *Topics in Polynomials: Extremal Problems, Inequalities, Zeros.* World Scientific, Singapore – New Jersey – London – Hong Kong, 1994.

[7] Q.I. Rahman and G. Schmeisser, *Inequalities for polynomials on the unit interval.* Trans. Amer. Math. Soc. **231** (1977), 93–100.

[8] I. Schur, *Über das Maximum des absoluten Betrages eines Polynoms in einem gegebenen Intervall.* Math. Z. **4** (1919), 217–287.

[9] Q.M. Tariq, *Concerning polynomials on the unit interval.* Proc. Amer. Math. Soc. **99** (1987), 293–296.

Gradimir V. Milovanović, Department of Mathematics, University of Niš, Faculty of Electronic Engineering, P.O. Box 73, 18000 Niš, Yugoslavia.